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Characterization of Morita Equivalence Pairs of Quantales

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We characterize the pairs of sup-lattices which occur as pairs of Morita equivalence bimodules between quantales in terms of the mutual relation between the sup-lattices.

KEY WORDS: quantale; involutive quantale; κ-quantale; module; bimodule; Morita equivalence; Morita context; Morita pair.

1. INTRODUCTION AND PRELIMINARIES

In this paper we give a characterization of those pairs (X, Y) of sup-lattices, which occur as pairs of Morita equivalence bimodules between m-regular quantales, in terms of the mutual relation between the sup-lattices X and Y. The sup-lattice characterization is based on a surprising similarity with the C^*-algebra theoretical characterization of Morita pairs given in (Todorov, 2002).

Our main motivation comes from both algebra and analysis, where the concept of Morita equivalence has been applied to many different categories to explore the relationship between objects and their representation theory, i.e., the theory of modules.

The idea of Morita equivalence was first made precise by Morita (1958) in the context of the category of unital rings: two unital rings are called *Morita equivalent* if their categories of right (left) modules are equivalent. Morita equivalent rings always come with a pair of corresponding bimodules of a certain type in such a way that the functors implementing the equivalence of the categories of modules are actually equivalent to tensoring with these modules. Morita equivalent rings share many ring theoretical properties—the Morita invariants.

The notion of Morita equivalence has been adapted to many other algebraic and analytic contexts such as nonunital rings (Ánh and Márki, 1987), rings with

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involution (Ara, 1999), C^*-algebras (Blecher *et al.*, 2000; Rieffel, 1974), monoids (Banaschewski, 1972), unital quantales (Borceux and Vitale, 1992) and nonunital involutive quantales (Paseka, 2001).

The paper is organized as follows. Section 1 introduces the basic notions concerning the subject. Section 2 contains the basic theorem for quantales, Section 3 its counterpart for involutive quantales. In Section 4, we show that our characterization results from Sections 1 and 2 translate to multiplicative (involutive) semilattices and, in fact, to (involutive) κ-quantales.

A *quantale* is a sup-lattice A with an associative binary multiplication satisfying

$$x \cdot \bigvee_{i \in I} x_i = \bigvee_{i \in I} x \cdot x_i \quad \text{and} \quad \left(\bigvee_{i \in I} x_i \right) \cdot x = \bigvee_{i \in I} x_i \cdot x$$

for all $x,\ x_i \in A,\ i \in I$ (I is a set). 1 denotes the greatest element of A, 0 is the smallest element of A. A quantale A is said to be *unital* if there is an element $e \in A$ such that $e \cdot a = a = a \cdot e$ for all $a \in A$.

An *involution* on a sup-lattice S is a unary operation such that

$$a^{**} = a,$$
$$\left(\bigvee a_i\right)^* = \bigvee a_i^*$$

for all $a, a_i \in S$. An *involution* on a quantale A is an involution on the sup-lattice A such that

$$(a \cdot b)^* = b^* \cdot a^*,$$

for all $a, b \in A$. A sup-lattice (quantale) with the involution is said to be *involutive*.

By a *morphism of* (*involutive*) *quantales* will be meant a $\bigvee-$ ($^*-$) and $\cdot$-preserving mapping $f : A \to A'$. If a morphism preserves the unital element we say that it is *unital*.

Let A be a quantale. A *right module over* A (shortly a right A-module) is a sup-lattice M, together with a *module action* $\_\cdot\_ : M \times A \to M$ satisfying

$$m \cdot (a \cdot b) = (m \cdot a) \cdot b \tag{M1}$$
$$\left(\bigvee X\right) \cdot a = \bigvee \{x \cdot a : x \in X\} \tag{M2}$$
$$m \cdot \bigvee S = \bigvee \{m \cdot s : s \in S\} \tag{M3}$$

for all $a, b \in A, m \in M, S \subseteq A, X \subseteq M$.

For a right A-module X the submodule $\mathrm{ess}(X) = X \cdot A$ generated by the elements $x \cdot a$ is called the *essential part* of X. If $\mathrm{ess}(X) = X$ we say that X is *essential*.

We shall say that A is *right separating* for the A-module M and that M is (*right*) *separated* by A if $m \cdot (-) = n \cdot (-)$ implies $m = n$. We say that M is *m-regular* if it is both separated by A and essential.

All definitions and propositions stated for right A-modules are valid in a dualized form for left A-modules. An (*m-regular*) A, B-*bimodule* is a sup-lattice M, together with a left (m-regular) A-module action and a right (m-regular) B-module action such that the left action with elements in A and the right action with elements in B commute.

An (involutive) quantale A is called *m-regular* if it is m-regular as an A, A-bimodule. Then, evidently $1 \cdot 1 = 1$ in A, $(-) \cdot a = (-) \cdot b$ implies $a = b$ and $a \cdot (-) = b \cdot (-)$ implies $a = b$.

For facts concerning quantales and quantale modules in general we refer to (Rosenthal, 1990).

To make our paper elementary and self-contained we will give an explicit and elementary definition of Morita equivalence of quantales (that is of course equivalent to the usual one as it was recently shown by the author (Paseka, 2002)).

Two m-regular quantales A and B are said to be *Morita equivalent* if there exist sup-lattices X and Y, such that

1) X is an m-regular A, B-bimodule, Y is an m-regular B, A-bimodule;
2) there are bimodule maps $(-, -) : X \times Y \longrightarrow A$ and $[-, -] : Y \times X \longrightarrow B$ called pairings such that $(x \cdot b, y) = (x, b \cdot y)$, $[y \cdot a, x] = [y, a \cdot x]$ (that is, these maps are *balanced*), $(x_1, y) \cdot x_2 = x_1 \cdot [y, x_2]$, $[y_1, x] \cdot y_2 = y_1 \cdot (x, y_2)$ for each $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$, $a \in A$, $b \in B$; and
3) the sup-preserving maps on the sup-lattice tensor products $X \otimes Y$, $Y \otimes X$ induced by the pairings are surjective.

The six-tuple $(A, B, X, Y, (-, -), [-, -])$ is called a *Morita context* and the pair (X, Y) a *Morita pair*.

If X is a sup-lattice, we denote by $\mathcal{Q}(X)$ the quantale of sup-preserving operators on X.

2. MORITA PAIRS BETWEEN QUANTALES

In this section we characterize Morita pairs between m-regular quantales.

Theorem 1. *Let X and Y be sup-lattices. Then (X, Y) is a Morita equivalence pair between some m-regular quantales if and only if there exist surjective sup-preserving maps $p : X \otimes Y \otimes X \to X$ and $q : Y \otimes X \otimes Y \to Y$ such that*

1. $p(p(x_1 \otimes y_1 \otimes x_2) \otimes y_2 \otimes x_3) = p(x_1 \otimes q(y_1 \otimes x_2 \otimes y_2) \otimes x_3)$ $= p(x_1 \otimes y_1 \otimes p(x_2 \otimes y_2 \otimes x_3))$;
2. $q(q(y_1 \otimes x_1 \otimes y_2) \otimes x_2 \otimes y_3) = q(y_1 \otimes p(x_1 \otimes y_2 \otimes x_2) \otimes y_3)$ $= q(y_1 \otimes x_1 \otimes q(y_2 \otimes x_2 \otimes y_3))$;
3. $p(- \otimes x_1) = p(- \otimes x_2)$ *implies* $x_1 = x_2$;
4. $p(x_1 \otimes -) = p(x_2 \otimes -)$ *implies* $x_1 = x_2$;

5. $q(- \otimes y_1) = q(- \otimes y_2)$ *implies* $y_1 = y_2$*; and*
6. $q(y_1 \otimes -) = q(y_2 \otimes -)$ *implies* $y_1 = y_2$

for each $x_i \in X$, $y_i \in Y$ *(*$i = 1, 2, 3$*).*

Proof: Assume that X and Y are sup-lattices, $p : X \otimes Y \otimes X \to X$ and $q : Y \otimes X \otimes Y \to Y$ are surjective sup-preserving maps which satisfy conditions 1–6.

We recall that $L_a : X \to X$ and $R_b : Y \to Y$ are the sup-preserving operators given by $L_a(x) = p(a \otimes x)$ and $R_b(y) = q(b \otimes y)$, where $a \in X \otimes Y$ and $b \in Y \otimes X$.

So we can think of $L_a(x)$ ($R_b(y)$) as a left module action by L_a (R_b) on X (Y). Identities 3) and 5) state that these actions are left separating.

Recall that $L_a \in \mathcal{Q}(X)$ and $R_b \in \mathcal{Q}(Y)$. Let $\mathcal{L} = \{L_a : a \in X \otimes Y\}$ and $\mathcal{R} = \{R_b : b \in Y \otimes X\}$. Evidently, $\mathcal{L}$ is a subsup-lattice of $\mathcal{Q}(X)$ and $\mathcal{R} = \{R_b : b \in Y \otimes X\}$ is a subsup-lattice of $\mathcal{Q}(Y)$.

We will only point out the "left" considerations in constructing the Morita context whenever the "right" ones follow either by symmetry or in a similar way.

First note that $\mathcal{L}$ is a subquantale of $\mathcal{Q}(X)$. Indeed, we have that

$$\begin{aligned} L_{x_1 \otimes y_1} L_{x_1 \otimes y_1}(z) &= p(x_1 \otimes y_1 \otimes p(x_2 \otimes y_2 \otimes z)) \\ &= p(p(x_1 \otimes y_1 \otimes x_2) \otimes y_2 \otimes z) \\ &= L_{p(x_1 \otimes y_1 \otimes x_2) \otimes y_2}(z) \end{aligned}$$

for each $z \in X$ and it follows that $L_a L_b \in \mathcal{L}$ whenever $a, b \in X \otimes Y$.

Next observe that $\mathcal{L}$ is m-regular according to 3) and the surjectivity of p. Indeed, if $x \in X$, $y \in Y$, then there are $x_i, u_i \in X$, $y_i \in Y$, $i \in I$ such that $x = \bigvee p(x_i \otimes y_i \otimes u_i)$. Then we have that

$$\bigvee_{i \in I} L_{x_i \otimes y_i} L_{u_i \otimes y} = L_{\bigvee_{i \in I} p(x_i \otimes y_i \otimes u_i) \otimes y} = L_{x \otimes y}.$$

So we have that $\mathcal{L}$ is an essential module over $\mathcal{L}$.

Now, let $a, b \in X \otimes Y$ and assume that $L_a L_{x \otimes y} = L_b L_{x \otimes y}$ for all $x \in X$ and all $y \in Y$. Then $L_a(L_{x \otimes y}(z)) = L_b(L_{x \otimes y}(z))$ for all $x, z \in X$ and all $y \in Y$, i.e., $L_a(p(x \otimes y \otimes z)) = L_b(p(x \otimes y \otimes z))$ for all $x, z \in X$ and all $y \in Y$, i.e., $L_a(u) = L_b(u)$ for all $u \in X$ because p is a surjective map.

Similarly, let us assume that $L_c L_a = L_c L_b$ for all $c \in X \otimes Y$. Then, for all $z \in X$, we have that $p(c \otimes L_a(z)) = p(c \otimes L_b(z))$, i.e., $L_a(z) = L_b(z)$ by the condition 3).

Altogether, $\mathcal{L}$ is an m-regular quantale.

Let us endow the sup-lattice X with the structure of an $\mathcal{L}, \mathcal{R}$-bimodule, by letting $L_a \cdot x = p(a \otimes x)$ and $x \cdot R_b = p(x \otimes b)$. In a similar way, we will endow the sup-lattice Y with the structure of an $\mathcal{R}, \mathcal{L}$-bimodule, by letting $R_b \cdot y = q(b \otimes y)$ and $y \cdot L_a = q(y \otimes a)$.

Define pairings $(-, -) : X \times Y \to \mathcal{L}$ and $[-, -] : Y \times X \to \mathcal{R}$ by setting $(x, y) = L_{x \otimes y}$ and $[y, x] = R_{y \otimes x}$, where $x \in X$ and $y \in Y$.

The fact that the pairings are balanced bimodule maps and the "linking" properties between the two pairings are verified easily.

For the converse direction, suppose that $(A, B, X, Y, (-, -), [-, -])$ is a Morita context. Define $\tilde{p} : X \times Y \times X \to X$ and $\tilde{q} : Y \times X \times Y \to Y$ by letting

$$\tilde{p}(x_1, y, x_2) = (x_1, y) \cdot x_2$$

and

$$\tilde{q}(y_1, x, y_2) = [y_1, x] \cdot y_2.$$

It is obvious that $\tilde{p}$ and $\tilde{q}$ are tri-sup-preserving maps.

If $p : X \otimes Y \otimes X \to X$ and $q : Y \otimes X \otimes Y \to Y$ are their extensions through the sup-lattice tensor product, then it is easily seen from the definition of a Morita context that conditions 1-6 hold (since they are valid on generators) and both p and q are surjective sup-preserving maps (since the maps $(-, -)$ and $[-, -]$ are surjections onto A and B and both X, Y are m-regular). □

3. THE INVOLUTIVE CASE

In this section we consider the consequences of Theorem 1 when the quantales in the Morita context are involutive. If X is a sup-lattice, then we are able to form its conjugate sup-lattice X^*: as a sup-lattice, we have $X^* = X$ and $^* : X \to X^*$, $x \mapsto x^*$ denotes the identity map.

To each given A, B-bimodule X, A, B involutive quantales, there naturally corresponds a B, A-bimodule X^* with a left B-action and a right A action on X^* given by

$$b \cdot x^* = (x \cdot b^*)^*, x^* \cdot a = (a^* \cdot x)^*$$

for all $a \in A$, $b \in B$.

Recall that an imprimitivity bimodule between two m-regular involutive quantales A and B (see Paseka, 2001) is a sup-lattice X which is an m-regular A,B-bimodule and which is equipped with inner products ${}_A\langle -, - \rangle : X \times X \to A$ and $\langle -, - \rangle_B : X \times X \to B$, such that $(A, X, {}_A\langle \cdot, \cdot \rangle)$ and $(B, X, \langle \cdot, \cdot \rangle_B)$ are full Hilbert modules and ${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$ - the compatibility condition.

If A and B are m-regular involutive quantales and X is an imprimitivity bimodule between A and B, then we are able to form an imprimitivity B, A-bimodule X^* in the natural way, letting ${}_B\langle x^*, y^* \rangle = \langle y, x \rangle_B^*$ and $\langle x^*, y^* \rangle_A = {}_A\langle y, x \rangle^*$.

Then, two m-regular involutive quantales A and B are Morita equivalent via a Morita context $(A, B, X, X^*, (-, -), [-, -])$ if and only if X is an imprimitivity bimodule between A and B. Namely, given an imprimitivity A, B-bimodule, letting $(x, y^*) =_A \langle x, y\rangle$, $[x^*, y] = \langle x, y\rangle_B$, the condition 2) from the definition of Morita equivalence holds by the compatibility condition. Since ${}_A\langle -, -\rangle$ and $\langle -, -\rangle_B$ are full, it follows that $(-, -)$ and $[-, -]$ are surjective. The converse direction goes the same way.

Theorem 2. *Let X be an sup-lattice. Then X is an imprimitivity bimodule between certain Morita equivalent m-regular involutive quantales if and only if there exists a surjective sup-preserving map $p : X \otimes X^* \otimes X \to X$ such that*

a) $p(p(x_1 \otimes x_2^* \otimes x_3) \otimes x_4^* \otimes x_5) = p(x_1 \otimes p(x_4 \otimes x_3^* \otimes x_2)^* \otimes x_5) = p(x_1 \otimes x_2^* \otimes p(x_3 \otimes x_4^* \otimes x_5))$;
b) $p(- \otimes x_1) = p(- \otimes x_2)$ *implies* $x_1 = x_2$; *and*
c) $p(x_1 \otimes -) = p(x_2 \otimes -)$ *implies* $x_1 = x_2$

for each $x_i \in X$ $(i = 1, 2, 3, 4, 5)$.

Proof: Suppose that a sup-lattice X and a map p are given such that a), b) and c) are satisfied. Let $q : X^* \otimes X \otimes X^* \to X^*$ be the map given by

$$q(x^* \otimes y \otimes z^*) = p(z \otimes y^* \otimes x)^*, \quad x, y, z \in X.$$

An immediate verification shows that p and q satisfy conditions 1–6 of Theorem 1.

Note that $\mathcal{L}$ and $\mathcal{R}$ are m-regular involutive quantales with respect to the involutions on $\mathcal{L}$ and $\mathcal{R}$ defined by: if $x, y \in X$, we let $L^*_{x\otimes y^*} = L_{y\otimes x^*}$ and $R^*_{y^*\otimes x} = R_{x^*\otimes y}$. The fact that the mappings $L \to L^*$ and $R \to R^*$ are indeed involutions on $\mathcal{L}$ and $\mathcal{R}$ follows easily from condition a).

Thus, we have shown that $(\mathcal{L}, \mathcal{R}, X, X^*, (-, -), [-, -])$ is a Morita context.

The converse direction is obtained similarly as the converse direction in Theorem 1. □

4. MORITA PAIRS BETWEEN κ-QUANTALES

The following definitions are either taken from (Nkuimi-Jugnia, 2000) or they are a straightforward reformulation of those in Section 1. In what follows we shall always assume that κ will be an infinite regular cardinal.

By a *κ-join semilattice* we understand a poset $(S, \leq)$ for which every subset of cardinality strictly less than κ has a join. Such a join is called a *κ-join*. *Morphisms* between κ-semilattices are required to preserve κ-joins. Note that any κ-join semilattice has arbitrary finite joins, especially it has the bottom element 0. If $\kappa = \omega$ we call ω-join semilattices only *join semilattices*.

By a *κ-quantale* is meant a κ-join semilattice equipped with an associative multiplication; this multiplication distributes over arbitrary κ-joins. A κ-quantale is said to be *unital* if the multiplication has a unit. If $\kappa = \omega$ we call ω-quantales *m-semilattices* (see Paseka, 1986).

An *involution* on a κ-join semilattice S is a unary operation such that

$$a^{**} = a,$$
$$(\bigvee_{i\in\kappa} a_i)^* = \bigvee_{i\in\kappa} a_i^*$$

for all $a, a_i \in S$, $i \in \kappa$. An *involution* on a κ-quantale A is an involution on the κ-join semilattice A such that

$$(a \cdot b)^* = b^* \cdot a^*,$$

for all $a, b \in A$. A κ-join semilattice (κ-quantale) with the involution is said to be *involutive*.

By a *morphism of* (*involutive*) *κ-quantales* will be meant a κ-joins- (*-) and $\cdot$-preserving mapping $f : A \to A'$. If a morphism preserves the unital element we say that it is *unital*.

Let A be a κ-quantale. A *right κ-module over A* (shortly a right A-κ-module) is a κ-join semilattice M, together with a *module action* $\_\cdot\_ : M \times A \to M$ satisfying

$$m \cdot (a \cdot b) = (m \cdot a) \cdot b \qquad \text{(M1)}$$
$$(\bigvee X) \cdot a = \bigvee\{x \cdot a : x \in X\} \qquad (\kappa\text{-M2})$$
$$m \cdot \bigvee S = \bigvee\{m \cdot s : s \in S\} \qquad (\kappa\text{-M3})$$

for all $a, b \in A$, $m \in M$, $S \subseteq A$, $|S| < \kappa$, $X \subseteq M$, $|X| < \kappa$. Morphism of κ-modules over a κ-quantale A are morphisms of κ-join semilattices which preserve the action by A.

By the same procedure as in Section 1 we may define the κ-essential part of a modules, right-separatedness, m-regularity and other basic notions from the quantale case. So we shall omit it.

Two m-regular κ-quantales A and B are said to be *Morita equivalent* if there exist κ-join semilattices X and Y, such that

1) X is an m-regular A, B-κ-bimodule, Y is an m-regular B, A-κ-bimodule;
2) there are κ-bimodule maps $(-,-) : X \times Y \longrightarrow A$ and $[-,-] : Y \times X \longrightarrow B$ called pairings such that $(x \cdot b, y) = (x, b \cdot y)$, $[y \cdot a, x] = [y, a \cdot x]$ (that is, these maps are *balanced*), $(x_1, y) \cdot x_2 = x_1 \cdot [y, x_2]$, $[y_1, x] \cdot y_2 = y_1 \cdot (x, y_2)$ for each $x, x_1, x_2 \in X$, $y, y_1, y_2 \in Y$, $a \in A$, $b \in B$ and
3) the κ-join-preserving maps on the κ-join semilattice tensor products $X \otimes_\kappa Y$, $Y \otimes_\kappa X$ induced by the pairings are surjective.

The six-tuple $(A, B, X, Y, (-,-), [-,-])$ is called a *Morita context* and the pair (X, Y) a *Morita pair*.

If X is a κ-join semilattice, we denote by $\mathcal{Q}_\kappa(X)$ the κ-quantale of κ-join-preserving operators on X.

Since all arguments in the proofs of the following propositions are the same as in Theorems 1 and 2, we shall omit them.

So we only state the corresponding theorems in the κ-setting.

Theorem 3. *Let X and Y be κ-join semilattices. Then (X, Y) is a Morita equivalence pair between some m-regular κ-quantales if and only if there exist surjective κ-join semilattice morphisms $p : X \otimes Y \otimes X \to X$ and $q : Y \otimes X \otimes Y \to Y$ satisfying conditions 1–6 from Theorem 1.*

Corollary 4. *Let X and Y be join semilattices. Then (X, Y) is a Morita equivalence pair between some m-regular m-semilattices if and only if there exist surjective join semilattice morphisms $p : X \otimes Y \otimes X \to X$ and $q : Y \otimes X \otimes Y \to Y$ satisfying conditions 1–6 from Theorem 1.*

Theorem 5. *Let X be a κ-join semilattice. Then (X, X^*) is a Morita equivalence pair between some involutive m-regular κ-quantales if and only if there exists a surjective κ-join semilattice morphism $p : X \otimes X^* \otimes X \to X$ satisfying conditions a)–c) from Theorem 2.*

Corollary 6. *Let X be a join semilattice. Then (X, X^*) is a Morita equivalence pair between some involutive m-regular m-semilattices if and only if there exists a surjective join semilattice morphism $p : X \otimes X^* \otimes X \to X$ satisfying conditions a)–c) from Theorem 2.*

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